

PLURICANONICAL SYSTEMS OF PROJECTIVE VARIETIES OF GENERAL TYPE

Hajime TSUJI

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Contents

1	Introduction	1
2	Multiplier ideal sheaves and singularities of divisors	4
3	Proof of Theorem 1.1 and 1.2 assuming MMP	6
3.1	Construction of a stratification	6
3.2	Construction of the stratification as a family	12
3.3	Use of Kawamata's subadjunction theorem	14
3.4	Estimate of the degree	16
3.5	Completion of the proof of Theorem 1.1 and 1.2 assuming MMP	17
4	Proof of Theorem 1.1 and 1.2 without assuming MMP	17
4.1	Analytic Zariski decomposition	18
4.2	The L^2 -extension theorem	19
4.3	A construction of the function Ψ	21
4.4	Volume of pseudoeffective line bundles	21
4.5	Construction of stratifications	22
4.6	Another subadjunction theorem	25
4.7	Positivity result	27
4.8	Use of two subadjunction theorems	28
4.9	Estimate of the degree	30
4.10	Completion of the proof of Theorem 1.1 and 1.2.	31
5	The Severi-Iitaka conjecture	32
6	Appendix	33
6.1	Volume of nef and big line bundles	33
6.2	A Serre type vanishing theorem	33

1 Introduction

Let X be a smooth projective variety and let K_X be the canonical bundle of X . X is said to be a general type., if there exists a positive integer m such that

the pluricanonical system $|mK_X|$ gives a birational (rational) embedding of X . The following problem is fundamental to study projective variety of general type.

Problem Let X be a smooth projective variety of general type. Find a positive integer m_0 such that for every $m \geq m_0$, $|mK_X|$ gives a birational rational map from X into a projective space.

If $\dim X = 1$, it is well known that $|3K_X|$ gives a projective embedding. In the case of smooth projective surfaces of general type, E. Bombieri showed that $|5K_X|$ gives a birational rational map from X into a projective space ([3]). But for the case of $\dim X \geq 3$, very little is known about the above problem.

The main purpose of this article is to prove the following theorems.

Theorem 1.1 *There exists a positive integer ν_n which depends only on n such that for every smooth projective n -fold X of general type defined over complex numbers, $|mK_X|$ gives a birational rational map from X into a projective space for every $m \geq \nu_n$.*

Theorem 1.1 is very much related to the theory of minimal models. It has been conjectured that for every nonuniruled smooth projective variety X , there exists a projective variety X_{min} such that

1. X_{min} is birationally equivalent to X ,
2. X_{min} has only \mathbf{Q} -factorial terminal singularities,
3. $K_{X_{min}}$ is a nef \mathbf{Q} -Cartier divisor.

X_{min} is called a minimal model of X . To construct a minimal model, the minimal model program (MMP) has been proposed (cf [15, p.96]). The minimal model program was completed in the case of 3-folds by S. Mori ([20]).

The proof of Theorem 1.1 can be very much simplified, if we assume the existence of minimal models for projective varieties of general type. The proof for the general case is modeled after the proof under the existence of minimal models by using the theory of AZD.

We should also note that even if we assume the existence of minimal models for projective varieties of general type, Theorem 1.1 is quite nontrivial because the index of a minimal model of X ([15, p.159, Definition 5.19]) can be arbitrarily large. Conversely if we assume the existence of a minimal model of X and bound the index of the minimal model of X , then the proof of Theorem 1.1 is almost trivial. Since the index of minimal 3-folds of general type is unbounded, Theorem 1.1 is quite nontrivial even in the case of $\dim X = 3$. Hence in this sense the major difficulty of the proof of Theorem 1.1 is to find “**a (universal) lower bound**” of the positivity of K_X . In fact Theorem 1.1 is equivalent to the following theorem.

Theorem 1.2 *There exists a positive number C_n which depends only on n such that for every smooth projective n -fold X of general type defined over complex*

numbers,

$$\mu(X, K_X) := n! \cdot \overline{\lim}_{m \rightarrow \infty} m^{-\dim X} \dim H^0(X, \mathcal{O}_X(mK_X)) \geq C_n$$

holds.

We note that $\mu(X, K_X)$ is equal to the intersection number K_X^n for a minimal projective n -fold X of general type (cf. Proposition 5.1 in Appendix). In Theorem 1.1 and 1.2, the numbers ν_n and C_n have not yet been computed effectively except for the case of $n = 3$ ([33]).

The relation of Theorem 1.1 and 1.2 is as follows. Theorem 1.2 means that there exists a universal lower bound of the positivity of canonical bundle of smooth projective variety of general type with fixed dimension. On the other hand, for a smooth projective variety of general type X , the lower bound of m such that $|mK_X|$ gives a birational embedding depends on the positivity of K_X on certain families of subvarieties which dominates X as in Section 3 below. These families appear as the strata of the stratification as in [30, 1] which are the center of the log canonical singularities.

The positivity of K_X on the subvarieties can be related to the positivity of the canonical bundles of the smooth model of the subvarieties via the subadjunction theorem due to Kawamata ([11]). We note that since the family dominates X , for a general point of X , all the members of the family passing through the point should be of general type.

The organization of the paper is as follows. In Section 2, we review the relation between multiplier ideal sheaves and singularity of divisors. In Section 3, we prove Theorem 1.1 and 1.2 assuming the existence of minimal models for projective varieties of general type. For the proof we use the induction on the dimension. Section 3.1 and 3.2 are similar to the argument as in [30, 1]. The essential part of Section 3 consists of Section 3.3 and 3.4. In Section 3.3, we use the subadjunction theorem of Kawamata to relate the canonical divisor of centers of log canonical singularities and the canonical divisor of ambient space. In Section 3.4, we prove that the minimal projective n -fold X of general type with $K_X^n \leq 1$ can be embedded birationally into a projective space as a variety with degree $\leq C^n$, where C is a positive constant depending only on n (for the definition of C , see Lemma 3.10). Using this fact we finish the proof of Theorem 1.1 and 1.2 (assuming the existence of minimal models).

In Section 4, we prove Theorem 1.1 and 1.2 without assuming the existence of minimal models for projective varieties of general type. Here we use the AZD (cf. Section 4.1) of K_X instead of minimal models. The only essential difference of Section 3 and 4 is the use of the another subadjunction theorem (cf. Section 4.6, 4.7, 4.8). The rest is nothing but the transcription of Section 3 using the intersection theory of singular hermitian line bundles.

In Section 5, we discuss the application of Theorem 1.1 and 1.2 to Severi-Iitaka's conjecture.

In this paper all the varieties are defined over \mathbf{C} .

2 Multiplier ideal sheaves and singularities of divisors

Before starting the proofs of Theorem 1.1 and 1.2, we shall review the relation between multiplier ideal sheaves and singularities of divisors. In this subsection L will denote a holomorphic line bundle on a complex manifold M .

Definition 2.1 *A singular hermitian metric h on L is given by*

$$h = e^{-\varphi} \cdot h_0,$$

where h_0 is a C^∞ -hermitian metric on L and $\varphi \in L^1_{loc}(M)$ is an arbitrary function on M . We call φ a weight function of h .

The curvature current Θ_h of the singular hermitian line bundle (L, h) is defined by

$$\Theta_h := \Theta_{h_0} + \sqrt{-1} \partial \bar{\partial} \varphi,$$

where $\partial \bar{\partial}$ is taken in the sense of a current. The L^2 -sheaf $\mathcal{L}^2(L, h)$ of the singular hermitian line bundle (L, h) is defined by

$$\mathcal{L}^2(L, h) := \{\sigma \in \Gamma(U, \mathcal{O}_M(L)) \mid h(\sigma, \sigma) \in L^1_{loc}(U)\},$$

where U runs over the open subsets of M . In this case there exists an ideal sheaf $\mathcal{I}(h)$ such that

$$\mathcal{L}^2(L, h) = \mathcal{O}_M(L) \otimes \mathcal{I}(h)$$

holds. We call $\mathcal{I}(h)$ the **multiplier ideal sheaf** of (L, h) . If we write h as

$$h = e^{-\varphi} \cdot h_0,$$

where h_0 is a C^∞ hermitian metric on L and $\varphi \in L^1_{loc}(M)$ is the weight function, we see that

$$\mathcal{I}(h) = \mathcal{L}^2(\mathcal{O}_M, e^{-\varphi})$$

holds. For $\varphi \in L^1_{loc}(M)$ we define the multiplier ideal sheaf of φ by

$$\mathcal{I}(\varphi) := \mathcal{L}^2(\mathcal{O}_M, e^{-\varphi}).$$

Example 2.1 *Let $\sigma \in \Gamma(X, \mathcal{O}_X(L))$ be the global section. Then*

$$h := \frac{1}{|\sigma|^2} = \frac{h_0}{h_0(\sigma, \sigma)}$$

is a singular hermitian metric on L , where h_0 is an arbitrary C^∞ -hermitian metric on L (the righthandside is obviously independent of h_0). The curvature Θ_h is given by

$$\Theta_h = 2\pi\sqrt{-1}(\sigma)$$

where (σ) denotes the current of integration over the divisor of σ .

Definition 2.2 L is said to be pseudoeffective, if there exists a singular hermitian metric h on L such that the curvature current Θ_h is a closed positive current.

Also a singular hermitian line bundle (L, h) is said to be pseudoeffective, if the curvature current Θ_h is a closed positive current.

If $\{\sigma_i\}$ are a finite number of global holomorphic sections of L , for every positive rational number α and a C^∞ -function ϕ ,

$$h := e^{-\phi} \cdot \frac{1}{\sum_i |\sigma_i|^{2\alpha}}$$

defines a singular hermitian metric on αL . We call such a metric h a singular hermitian metric on αL with **algebraic singularities**. Singular hermitian metrics with algebraic singularities are particularly easy to handle, because its multiplier ideal sheaf of the metric can be controlled by taking suitable successive blowing ups such that the total transform of the divisor $\sum_i (\sigma_i)$ is a divisor with normal crossings.

Let $D = \sum a_i D_i$ be an effective \mathbf{R} -divisor on X . Let σ_i be a section of $\mathcal{O}_X(D_i)$ with divisor D_i respectively. Then we define

$$\mathcal{I}(D) := \mathcal{I}\left(\prod_i \frac{1}{|\sigma_i|^{2a_i}}\right)$$

and call it the multiplier ideal of the divisor D .

Let us consider the relation between $\mathcal{I}(D)$ and singularities of D .

Definition 2.3 Let X be a normal variety and $D = \sum_i d_i D_i$ an effective \mathbf{Q} -divisor such that $K_X + D$ is \mathbf{Q} -Cartier. If $\mu : Y \rightarrow X$ is an embedded resolution of the pair (X, D) , then we can write

$$K_Y + \mu_*^{-1} D = \mu^*(K_X + D) + F$$

with $F = \sum_j e_j E_j$ for the exceptional divisors $\{E_j\}$. We call F the discrepancy and $e_j \in \mathbf{Q}$ the discrepancy coefficient for E_j . We regard $-d_i$ as the discrepancy coefficient of D_i .

The pair (X, D) is said to have only **Kawamata log terminal singularities** (KLT) (resp. **log canonical singularities** (LC)), if $d_i < 1$ (resp. ≤ 1) for all i and $e_j > -1$ (resp. ≥ -1) for all j for an embedded resolution $\mu : Y \rightarrow X$. One can also say that (X, D) is KLT (resp. LC), or $K_X + D$ is KLT (resp. LC), when (X, D) has only KLT (resp. LC). The pair (X, D) is said to be KLT (resp. LC) at a point $x_0 \in X$, if $(U, D|_U)$ is KLT (resp. LC) for some neighbourhood U of x_0 .

The following proposition is a dictionary between algebraic geometry and the L^2 -method.

Proposition 2.1 Let D be a divisor on a smooth projective variety X . Then (X, D) is KLT, if and only if $\mathcal{I}(D)$ is trivial ($= \mathcal{O}_X$).

The proof is trivial and left to the readers. To locate the co-support of the multiplier ideal the following notion is useful.

Definition 2.4 *A subvariety W of X is said to be a **center of log canonical singularities** for the pair (X, D) , if there is a birational morphism from a normal variety $\mu : Y \rightarrow X$ and a prime divisor E on Y with the discrepancy coefficient $e \leq -1$ such that $\mu(E) = W$.*

The set of all the centers of log canonical singularities is denoted by $CLC(X, D)$. For a point $x_0 \in X$, we define $CLC(X, x_0, D) := \{W \in CLC(X, D) \mid x_0 \in W\}$. We quote the following proposition to introduce the notion of the minimal center of logcanonical singularities.

Proposition 2.2 *([12, p.494, Proposition 1.5]) Let X be a normal variety and D an effective \mathbf{Q} -Cartier divisor such that $K_X + D$ is \mathbf{Q} -Cartier. Assume that X is KLT and (X, D) is LC. If $W_1, W_2 \in CLC(X, D)$ and W an irreducible component of $W_1 \cap W_2$, then $W \in CLC(X, D)$. In particular if (X, D) is not KLT at a point $x_0 \in X$, then there exists the unique minimal element of $CLC(X, x_0, D)$.*

We call the minimal element the **minimal center of LC singularities** of (X, D) at x_0 .

3 Proof of Theorem 1.1 and 1.2 assuming MMP

In this section we prove Theorem 1.1 and 1.2 assuming the minimal model program. The reason is that we can avoid inessential technicalities under this assumption. The full proof of Theorem 1.1 and 1.2 is essentially nothing but the transcription of the proof in this section by using the theory of AZD except the use of another subadjunction theorem (cf. Section 3.6, 3.8). Since the minimal model program is established in the case of 3-folds, the proof under this assumption provides the full proofs of Theorem 1.1 and 1.2 for the case of projective varieties of general type of $\dim X \leq 3$.

Let X be a minimal projective n -fold of general type, i.e., X has only \mathbf{Q} -factorial terminal singularities and the canonical divisor K_X is nef. We set

$$X^\circ = \{x \in X_{reg} \mid x \notin Bs \mid mK_X \mid \text{ and } \Phi_{|mK_X|} \text{ is a biholomorphism} \\ \text{on a neighbourhood of } x \text{ for some } m \geq 1\}.$$

Then X° is a nonempty Zariski open subset of X .

3.1 Construction of a stratification

The construction of a stratification below is similar to that in [30, 1]. The only difference is the fact that we deal with the \mathbf{Q} -Cartier divisor K_X which is not Cartier in general. Of course this difference is very minor.

We set

$$\mu_0 := K_X^n.$$

Lemma 3.1 *Let x, x' be distinct points on X° . We set*

$$\mathcal{M}_{x, x'} = \mathcal{M}_x \otimes \mathcal{M}_{x'},$$

where $\mathcal{M}_x, \mathcal{M}_{x'}$ denote the maximal ideal sheaves of the points x, x' respectively. Let ε be a sufficiently small positive number. Then

$$H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{M}_{x, x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon) \frac{m}{\sqrt[n]{2}} \rceil}) \neq 0$$

for every sufficiently large m .

Proof of Lemma 3.1. Let us consider the exact sequence:

$$0 \rightarrow H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{M}_{x, x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon) \frac{m}{\sqrt[n]{2}} \rceil}) \rightarrow H^0(X, \mathcal{O}_X(mK_X)) \rightarrow H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{O}_X / \mathcal{M}_{x, x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon) \frac{m}{\sqrt[n]{2}} \rceil}).$$

We note that

$$n! \cdot \overline{\lim}_{m \rightarrow \infty} m^{-n} \dim H^0(X, \mathcal{O}_X(mK_X)) = \mu_0$$

holds, since K_X is nef and big (cf. Proposition 5.1 in Appendix). Since

$$n! \cdot \overline{\lim}_{m \rightarrow \infty} m^{-n} \dim H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{O}_X / \mathcal{M}_{x, x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon) \frac{m}{\sqrt[n]{2}} \rceil}) = \mu_0(1-\varepsilon)^n < \mu_0$$

hold, we see that Lemma 3.1 holds. **Q.E.D.**

Let us take a sufficiently large positive integer m_0 and let σ be a general (nonzero) element σ_0 of $H^0(X, \mathcal{O}_X(m_0K_X) \otimes \mathcal{M}_{x, x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon) \frac{m_0}{\sqrt[n]{2}} \rceil})$. We define an effective **Q**-divisor D_0 by

$$D_0 = \frac{1}{m_0}(\sigma_0).$$

We define a positive number α_0 by

$$\alpha_0 := \inf\{\alpha > 0 \mid (X, \alpha D_0) \text{ is not KLT at both } x \text{ and } x'\},$$

where KLT is short for Kawamata log terminal (cf. Definition 2.3). Since $(\sum_{i=1}^n |z_i|^2)^{-n}$ is not locally integrable around $O \in \mathbf{C}^n$, by the construction of D_0 , we see that

$$\alpha_0 \leq \frac{n \sqrt[n]{2}}{\sqrt[n]{\mu_0}(1-\varepsilon)}$$

holds. About the relation between KLT condition and the multiplier ideal sheaves, please see Section 2.

Let us fix a positive number $\delta < 1$. If we take $\varepsilon > 0$ sufficiently small, we may and do assume that

$$\alpha_0 \leq \frac{n \sqrt[n]{2}}{\sqrt[n]{\mu_0}} + \delta$$

holds. Then one of the following two cases occurs.

Case 1.1: For every small positive number λ , $(X, (\alpha_0 - \lambda)D_0)$ is KLT at both x and x' .

Case 1.2: For every small positive number δ , $(X, (\alpha_0 - \lambda)D_0)$ is KLT at one of x or x' say x .

We first consider Case 1.1. Let X_1 be the minimal center of log canonical singularities at x (cf. Section 2). We consider the following two cases.

Case 3.1: X_1 passes through both x and x' ,
Case 3.2: Otherwise

First we consider Case 3.1. In this case X_1 is not isolated at x . Let n_1 denote the dimension of X_1 . Let us define the volume μ_1 of X_1 with respect to K_X by

$$\mu_1 := K_X^{n_1} \cdot X_1.$$

Since $x \in X^\circ$, we see that $\mu_1 > 0$ holds. The proof of the following lemma is identical to that of Lemma 3.1.

Lemma 3.2 *Let ε be a sufficiently small positive number and let x_1, x_2 be distinct regular points on X_1 . Then for a sufficiently large $m > 1$,*

$$H^0(X_1, \mathcal{O}_{X_1}(mK_X) \otimes \mathcal{M}_{x_1, x_2}^{\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon) \frac{m}{n_1 \sqrt{2}}} \rceil}) \neq 0$$

holds.

Let x_1, x_2 be two distinct regular points on $X_1 \cap X^\circ$. Let m_1 be a sufficiently large positive integer and Let

$$\sigma'_1 \in H^0(X_1, \mathcal{O}_{X_1}(m_1 K_X) \otimes \mathcal{M}_{x_1, x_2}^{\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon) \frac{m_1}{n_1 \sqrt{2}}} \rceil})$$

be a nonzero element.

By Kodaira's lemma there is an effective \mathbf{Q} -divisor E such that $K_X - E$ is ample. By the definition of X° , we may assume that the support of E does not contain both x and x' . Let ℓ_1 be a sufficiently large positive integer which will be specified later such that

$$L_1 := \ell_1(K_X - E)$$

is Cartier.

Lemma 3.3 *If we take ℓ_1 sufficiently large, then*

$$\phi_m : H^0(X, \mathcal{O}_X(mK_X + L_1)) \rightarrow H^0(X_1, \mathcal{O}_{X_1}(mK_X + L_1))$$

is surjective for every $m \geq 0$.

Proof. K_X is nef \mathbf{Q} -Cartier divisor by the assumption. Let r be the index of X , i.e. r is the minimal positive integer such that rK_X is Cartier. Then for every locally free sheaf \mathcal{E} , by Lemma 5.1 in Appendix, there exists a positive integer k_0 depending on \mathcal{E} such that for every $\ell \geq k_0$

$$H^q(X, \mathcal{O}_X((1 + mr)K_X + L_1) \otimes \mathcal{E}) = 0$$

holds for every $q \geq 1$ and $m \geq 0$. Let us consider the exact sequences

$$0 \rightarrow \mathcal{K}_j \rightarrow \mathcal{E}_j \rightarrow \mathcal{O}_X(jK_X) \otimes \mathcal{I}_{X_1} \rightarrow 0$$

for some locally free sheaf \mathcal{E}_j for every $0 \leq j \leq r-1$, where \mathcal{I}_{X_1} denotes the ideal sheaf associated with X_1 . Then noting the above fact, we can prove that if we take ℓ_1 sufficiently large,

$$H^q(X, \mathcal{O}_X(mK_X + L_1) \otimes \mathcal{I}_{X_1}) = 0$$

holds for every $q \geq 1$ and $m \geq 0$ by exactly the same manner as the standard proof of Serre's vanishing theorem (cf. [9, p.228, Theorem 5.2]). This implies the desired surjection. **Q.E.D.**

Let τ be a general section in $H^0(X, \mathcal{O}_X(L_1))$. Then by Lemma 3.3 we see that

$$\sigma'_1 \otimes \tau \in H^0(X_1, \mathcal{O}_{X_1}(m_1K_X + L_1) \mathcal{M}_{x_1, x_2}^{\lceil \frac{n\sqrt{\mu_1}(1-\varepsilon)}{n\sqrt{2}} \rceil})$$

extends to a section

$$\sigma_1 \in H^0(X, \mathcal{O}_X((m_1 + \ell_1)K_X)).$$

We may assume that there exists a neighbourhood $U_{x, x'}$ of $\{x, x'\}$ such that the divisor (σ_1) is smooth on $U_{x, x'} - X_1$ by Bertini's theorem, if we take ℓ_1 sufficiently large, since as in the proof of Lemma 3.3

$$H^0(X, \mathcal{O}_X(mK_X + L_1)) \rightarrow H^0(X, \mathcal{O}_X(mK_X + L_1) \otimes \mathcal{O}_X/\mathcal{I}_{X_1} \cdot \mathcal{M}_y)$$

is surjective for every $y \in X$ and $m \geq 0$. We set

$$D_1 = \frac{1}{m_1 + \ell_1}(\sigma_1).$$

Suppose that x, x' are **nonsingular points** on X_1 . Then we set $x_1 = x, x_2 = x'$. Let ε_0 be a sufficiently small positive rational number and define α_1 by

$$\alpha_1 := \inf\{\alpha > 0 \mid (\alpha_0 - \varepsilon_0)D_0 + \alpha D_1 \text{ is not KLT at both } x \text{ and } x'\}.$$

Then we may define the proper subvariety X_2 of X_1 as a minimal center of log canonical singularities as before.

Lemma 3.4 *Let δ be the fixed positive number as above, then we may assume that*

$$\alpha_1 \leq \frac{n_1 \frac{n\sqrt{2}}{n\sqrt{\mu_1}}}{\frac{n\sqrt{\mu_1}}{n\sqrt{\mu_1}}} + \delta$$

holds, if we make ε_0 and ℓ_1/m_1 sufficiently small.

To prove Lemma 3.4, we need the following elementary lemma.

Lemma 3.5 ([30, p.12, Lemma 6]) *Let a, b be positive numbers. Then*

$$\int_0^1 \frac{r_2^{2n_1-1}}{(r_1^2 + r_2^{2a})^b} dr_2 = r_1^{\frac{2n_1}{a}-2b} \int_0^{r_1^{-2a}} \frac{r_3^{2n_1-1}}{(1 + r_3^{2a})^b} dr_3$$

holds, where

$$r_3 = r_2/r_1^{1/a}.$$

Proof of Lemma 3.4. Let (z_1, \dots, z_n) be a local coordinate on a neighbourhood U of x in X such that

$$U \cap X_1 = \{q \in U \mid z_{n_1+1}(q) = \dots = z_n(q) = 0\}.$$

We set $r_1 = (\sum_{i=n_1+1}^n |z_i|^2)^{1/2}$ and $r_2 = (\sum_{i=1}^{n_1} |z_i|^2)^{1/2}$. Fix an arbitrary C^∞ -hermitian metric h_X on K_X . Then there exists a positive constant C such that

$$\|\sigma_1\|^2 \leq C(r_1^2 + r_2^{2\lceil \sqrt[n]{\mu_1}(1-\varepsilon)^{\frac{n_1}{n_1+2}} \rceil})$$

holds on a neighbourhood of x , where $\|\cdot\|$ denotes the norm with respect to $h_X^{m_1+\ell_1}$. We note that there exists a positive integer M such that

$$\|\sigma_1\|^{-2} = O(r_1^{-M})$$

holds on a neighbourhood of the generic point of $U \cap X_1$, where $\|\cdot\|$ denotes the norm with respect to $h_X^{m_0}$. Then by Lemma 3.5, we have the inequality

$$\alpha_1 \leq \left(\frac{m_1 + \ell_1}{m_1}\right) \frac{n_1 \sqrt[n]{2}}{\sqrt[n]{\mu_1}} + m_1 \varepsilon_0$$

holds. Taking ε_0 and ℓ_1/m_1 sufficiently small, we obtain that

$$\alpha_1 \leq \frac{n_1 \sqrt[n]{2}}{\sqrt[n]{\mu_1}} + \delta$$

holds. **Q.E.D.**

If x or x' is a singular point on X_1 , we need the following lemma.

Lemma 3.6 *Let φ be a plurisubharmonic function on $\Delta^n \times \Delta$. Let $\varphi_t (t \in \Delta)$ be the restriction of φ on $\Delta^n \times \{t\}$. Assume that $e^{-\varphi_t}$ does not belong to $L_{loc}^1(\Delta^n, O)$ for every $t \in \Delta^*$.*

Then $e^{-\varphi_0}$ is not locally integrable at $O \in \Delta^n$.

Lemma 3.6 is an immediate consequence of the L^2 -extension theorem [23, p.20, Theorem]. Using Lemma 3.6 and Lemma 3.5, we see that Lemma 3.4 holds by letting $x_1 \rightarrow x$ and $x_2 \rightarrow x'$.

Next we consider Case 1.2 and Case 3.2. In this case for every sufficiently small positive number δ , $(X, (\alpha_0 - \varepsilon_0)D_0 + (\alpha_1 - \delta)D_1)$ is KLT at x and not KLT at x' .

In these cases, instead of Lemma 3.2, we use the following simpler lemma.

Lemma 3.7 *Let ε be a sufficiently small positive number and let x_1 be a smooth point on X_1 . Then for a sufficiently large $m > 1$,*

$$H^0(X_1, \mathcal{O}_{X_1}(mK_X) \otimes \mathcal{M}_{x_1}^{\lceil \sqrt[n]{\mu_1}(1-\varepsilon)m \rceil}) \neq 0$$

holds.

Then taking a general nonzero element σ'_1 in

$$H^0(X_1, \mathcal{O}_{X_1}(m_1 K_X) \otimes \mathcal{I}(h^{m_1}) \otimes \mathcal{M}_{x_1}^{\lceil n\sqrt{\mu_1}(1-\varepsilon)m_1 \rceil}),$$

for a sufficiently large m_1 . As in Case 1.1 and Case 3.1 we obtain the proper subvariety X_2 in X_1 also in this case.

Inductively for distinct points $x, x' \in X^\circ$, we construct a strictly decreasing sequence of subvarieties

$$X = X_0 \supset X_1 \supset \cdots \supset X_r \supset X_{r+1} = x \text{ or } x'$$

and invariants (depending on small positive rational numbers $\varepsilon_0, \dots, \varepsilon_{r-1}$, large positive integers m_0, m_1, \dots, m_r , etc.) :

$$\alpha_0, \alpha_1, \dots, \alpha_r,$$

$$\mu_0, \mu_1, \dots, \mu_r$$

and

$$n > n_1 > \cdots > n_r.$$

By Nadel's vanishing theorem ([21, p.561]) we have the following lemma.

Lemma 3.8 *Let x, x' be two distinct points on X° . Then for every $m \geq \lceil \sum_{i=0}^r \alpha_i \rceil + 1$, $\Phi_{|mK_X|}$ separates x and x' . And we may assume that*

$$\alpha_i \leq \frac{n_i \sqrt[n_i]{2}}{\sqrt[n_i]{\mu_i}} + \delta$$

holds for every $0 \leq i \leq r$.

Proof. For $i = 0, 1, \dots, r$ let h_i be the singular hermitian metric on K_X defined by

$$h_i := \frac{1}{|\sigma_i|^{\frac{2}{m_i + \ell_i}}},$$

where we have set $\ell_0 := 0$. More precisely for any C^∞ -hermitian metric h_X on K_X we have defined h_i as

$$h_i := \frac{h_X}{h_X^{m_i + \ell_i}(\sigma_i, \sigma_i)^{\frac{1}{m_i + \ell_i}}}.$$

Using Kodaira's lemma ([14, Appendix]), let E be an effective \mathbf{Q} -divisor E such that $K_X - E$ is ample. Let m be a positive integer such that $m \geq \lceil \sum_{i=0}^r \alpha_i \rceil + 1$ holds. Let h_L is a C^∞ -hermitian metric on the ample \mathbf{Q} -line bundle

$$L := (m - 1 - (\sum_{i=0}^{r-1} (\alpha_i - \varepsilon_i)) - \alpha_r - \delta_L)K_X - \delta_L E$$

with strictly positive curvature, where δ_L be a sufficiently small positive number and we have considered h_L as a singular hermitian metric on $(m - 1 - (\sum_{i=0}^{r-1} (\alpha_i - \varepsilon_i)) - \alpha_r)K_X$. Let us define the singular hermitian metric $h_{x, x'}$ of $(m - 1)K_X$ defined by

$$h_{x, x'} = (\prod_{i=0}^{r-1} h_i^{\alpha_i - \varepsilon_i}) \cdot h_r^{\alpha_r} \cdot h_L.$$

Then we see that $\mathcal{I}(h_{x,x'})$ defines a subscheme of X with isolated support around x or x' by the definition of the invariants $\{\alpha_i\}$'s. By the construction the curvature current $\Theta_{h_{x,x'}}$ is strictly positive on X . Then by Nadel's vanishing theorem ([21, p.561]) we see that

$$H^1(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h_{x,x'})) = 0$$

holds. Hence

$$H^0(X, \mathcal{O}_X(mK_X)) \rightarrow H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{O}_X/\mathcal{I}(h_{x,x'}))$$

is surjective. Since by the construction of $h_{x,x'}$ (if we take δ_L sufficiently small) $\text{Supp}(\mathcal{O}_X/\mathcal{I}(h_{x,x'}))$ contains both x and x' and is isolated at least one of x or x' . Hence by the above surjection, there exists a section $\sigma \in H^0(X, \mathcal{O}_X(mK_X))$ such that

$$\sigma(x) \neq 0, \sigma(x') = 0$$

or

$$\sigma(x) = 0, \sigma(x') \neq 0$$

holds. This implies that $\Phi_{|mK_X|}$ separates x and x' .

The proof of the last statement is similar to the proof of Lemma 3.4. **Q.E.D.**

3.2 Construction of the stratification as a family

In this subsection we shall construct the above stratification as a family. But this is not absolutely necessary for our proof of Theorem 1.1 and 1.2. Please see Section 4.9 below for an alternative proof which bypasses this construction.

We note that for a fixed pair $(x, x') \in X^\circ \times X^\circ - \Delta_X$, $\sum_{i=0}^r \alpha_i$ depends on the choice of $\{X_i\}$'s, where Δ_X denotes the diagonal of $X \times X$. Moving (x, x') in $X^\circ \times X^\circ - \Delta_X$, we shall consider the above operation simultaneously. Let us explain the procedure. We set

$$B := X^\circ \times X^\circ - \Delta_X.$$

Let

$$p : X \times B \longrightarrow X$$

be the first projection and let

$$q : X \times B \longrightarrow B$$

be the second projection. Let Z be the subvariety of $X \times B$ defined by

$$Z := \{(x_1, x_2, x_3) : X \times B \mid x_1 = x_2 \text{ or } x_1 = x_3\}.$$

In this case we consider

$$q_* \mathcal{O}_{X \times B}(m_0 p^* K_X) \otimes \mathcal{I}_Z^{\lceil \sqrt[m_0]{\mu_0}(1-\varepsilon) \frac{m_0}{\sqrt{2}} \rceil}$$

instead of

$$H^0(X, \mathcal{O}_X(m_0 K_X) \otimes \mathcal{M}_{x,x'}^{\lceil \sqrt[m_0]{\mu_0}(1-\varepsilon) \frac{m_0}{\sqrt{2}} \rceil}),$$

where \mathcal{I}_Z denotes the ideal sheaf of Z . Let $\tilde{\sigma}_0$ be a nonzero global meromorphic section of

$$q_*\mathcal{O}_{X \times B}(m_0 p^* K_X) \otimes \mathcal{I}_Z^{\lceil \sqrt[m_0]{\mu_0}(1-\varepsilon) \frac{m_0}{\sqrt{2}} \rceil}$$

on B for a sufficiently large positive integer m_0 . We shall identify $\tilde{\sigma}_0$ with the family of sections of $m_0 p^* K_X$. We set

$$\tilde{D}_0 := \frac{1}{m_0}(\tilde{\sigma}_0).$$

We define the singular hermitian metric \tilde{h}_0 on $p^* K_X$ by

$$\tilde{h}_0 := \frac{1}{|\tilde{\sigma}_0|^{2/m_0}}.$$

We shall replace α_0 by

$$\tilde{\alpha}_0 := \inf\{\alpha > 0 \mid \text{the generic point of } Z \subseteq \text{Spec}(\mathcal{O}_{X \times B}/\mathcal{I}(h_0^\alpha))\}.$$

Then for every $0 < \delta \ll 1$, there exists a Zariski open subset U of B such that for every $b \in U$, $\tilde{h}_0|_{X \times \{b\}}$ is well defined and

$$b \not\subseteq \text{Spec}(\mathcal{O}_{X \times \{b\}}/\mathcal{I}(\tilde{h}_0^{\alpha_0 - \delta}|_{X \times \{b\}})),$$

where we have identified b with distinct two points in X . And also by Lemma 3.6, we see that

$$b \subseteq \text{Spec}(\mathcal{O}_{X \times \{b\}}/\mathcal{I}(\tilde{h}_0^{\alpha_0}|_{X \times \{b\}})),$$

holds for every $b \in B$. Let \tilde{X}_1 be the minimal center of log canonical singularities of $(X \times B, \alpha_0 \tilde{D}_0)$ at the generic point of Z . (although \tilde{D}_0 may not be effective this is meaningful by the construction of $\tilde{\sigma}_0$). We note that $\tilde{X}_1 \cap q^{-1}(b)$ may not be irreducible even for a general $b \in B$. But if we take a suitable finite cover

$$\phi_0 : B_0 \longrightarrow B,$$

on the base change $X \times_B B_0$, \tilde{X}_1 defines a family of irreducible subvarieties

$$f_1 : \hat{X}_1 \longrightarrow U_0$$

of X parametrized by a nonempty Zariski open subset U_0 of $\phi_0^{-1}(U)$. Let n_1 be the relative dimension of f_1 . We set

$$\tilde{\mu}_1 := K_X^{n_1} \cdot f_1^{-1}(b_0)$$

where b_0 is a general point on U_0 . Continuing this process we may construct a finite morphism

$$\phi_r : B_r \longrightarrow B$$

and a nonempty Zariski open subset U_r of B_r which parametrizes a family of stratification

$$X \supset X_1 \supset X_2 \supset \cdots \supset X_r \supset X_{r+1} = x \text{ or } x'$$

constructed as before. And we also obtain invariants $\{\tilde{\alpha}_0, \dots, \tilde{\alpha}_r\}$, $\{\tilde{\mu}_0, \dots, \tilde{\mu}_r\}$, $\{n = \tilde{n}_0, \dots, \tilde{n}_r\}$. Hereafter we denote these invariants without \sim for simplicity. By the same proof as in Lemma 3.4, we have the following lemma.

Lemma 3.9 *We may assume that*

$$\alpha_i \leq \frac{n_i \sqrt[n_i]{2}}{\sqrt[n_i]{\mu_i}} + \delta$$

holds for every $0 \leq i \leq r$.

By Lemma 3.8, we obtain that For every

$$m > \lceil \sum_{i=0}^r \alpha_i \rceil + 1$$

$|mK_X|$ separates points on the nonempty Zariski open subset $\phi_r(B_r)$.

Let us consider the complement $X^\circ - \phi_r(B_r)$. Replacing B by

$$(X^\circ - \phi_r(B_r)) \times (X^\circ - \phi_r(B_r)) - \Delta_X,$$

we may continue the same process. Hence by Noetherian induction, we have the following proposition.

Proposition 3.1 *There exists a finite stratification of $X \times X - \Delta_X$ such that the each stratum supports a (multivalued) family of stratification of X :*

$$X \supset X_1 \supset X_2 \supset \cdots \supset X_r \supset X_{r+1} = x \text{ or } x'$$

with the same invariants $\{\alpha_0, \dots, \alpha_r\}$, $\{\mu_0, \dots, \mu_r\}$ etc. (r may depend on the strata).

3.3 Use of Kawamata's subadjunction theorem

The following subadjunction theorem is crucial in our proof.

Theorem 3.1 ([11]) *Let X be a normal projective variety. Let D° and D be effective \mathbf{Q} -divisor on X such that $D^\circ < D$, (X, D°) is log terminal and (X, D) is log canonical. Let W be a minimal center of log canonical singularities for (X, D) . Let H be an ample Cartier divisor on X and ϵ a positive rational number. Then there exists an effective \mathbf{Q} -divisor D_W on W such that*

$$(K_X + D + \epsilon H) |_W \sim_{\mathbf{Q}} K_W + D_W$$

and (W, D_W) is log terminal. In particular W has only rational singularities.

Remark 3.1 *As is stated in [13, Remark 3.2], the assumption that W is a minimal center can be replaced that W is a local minimal center, since the argument in [11] which uses the variation of Hodge structure does not change. But in this case we need to replace K_W by the pushforward of the canonical divisor of the normalization of W , since W may be nonnormal ([12, p.494, Theorem 1.6] works only locally in this case).*

Roughly speaking, Theorem 3.1 implies that $K_X + D |_W$ (almost) dominates K_W .

Let us consider again the sequence of numbers α_j , divisors D_j and the stratification $X \supset X_1 \supset \cdots \supset X_j$ which were defined in Section 3.1. Let W_j be a nonsingular model of X_j . Applying Theorem 3.1 to $K_X + D$ where

$$D = (\alpha_0 - \varepsilon_0)D_0 + \cdots + (\alpha_{j-2} - \varepsilon_{j-2})D_{j-2} + \alpha_{j-1}D_{j-1},$$

we get

$$\mu(W_j, K_{W_j}) \leq \mu(W_j, K_{W_j} + D_{W_j}) \leq (1 + \sum_{i=0}^{j-1} \alpha_i)^{n_j} \cdot \mu_j$$

hold, where

$$\mu(W_j, K_{W_j}) := n_j! \cdot \overline{\lim}_{m \rightarrow \infty} m^{-n_j} \dim H^0(W_j, \mathcal{O}_{W_j}(mK_{W_j})).$$

We note that if we take x, x' general, W_j ought to be of general type. More precisely there exists no subfamily \mathcal{K} of $\{W_j\}$ (parametrized a quasiprojective variety) such that

1. every member of \mathcal{K} is of non-general type,
2. the members of \mathcal{K} dominates X by the natural morphism.

Otherwise X is dominated by a family of varieties of nongeneral type and this contradicts the assumption that X is of general type. Hence there exists a nonempty Zariski open set U_0 of X° such that if $(x, x') \in U_0 \times U_0$, then W_j is of general type for every j .

We shall prove Theorem 1.2 by induction on n . Suppose that Theorem 1.2 holds for projective varieties of general type of dimension less than or equal to $n - 1$ (the case of $n = 1$ is trivial), i.e., for every positive integer $k < n$ there exists a positive number $C(k)$ such that for every smooth projective variety W of general type of dimension k ,

$$\mu(W, K_W) \geq C(k)$$

holds.

Let us consider again the sequence of numbers α_j , divisors D_j and the stratification $X \supset X_1 \supset \cdots \supset X_j$ which were defined in Section 3.1. Then by the above inequality

$$C(n_j) \leq (1 + \sum_{i=0}^{j-1} \alpha_i)^{n_j} \cdot \mu_j$$

holds. Since

$$\alpha_i \leq \frac{\sqrt[n_i]{2} n_i}{\sqrt[n_i]{\mu_i}} + \delta$$

holds by Lemma 3.9, we see that

$$\frac{1}{\sqrt[n_j]{\mu_j}} \leq (1 + \sum_{i=0}^{j-1} \frac{\sqrt[n_i]{2} n_i}{\sqrt[n_i]{\mu_i}}) \cdot C(n_j)^{-\frac{1}{n_j}}$$

holds for every $j \geq 1$. We recall the finite stratification of $X^\circ \times X^\circ - \Delta_X$ in Section 3.2. Using the above inequality inductively, we have the following lemma.

Lemma 3.10 *Suppose that $\mu_0 \leq 1$ holds. Then there exists a positive constant C depending only on n such that for every $(x, x') \in U_0 \times U_0 - \Delta_X$ the corresponding invariants $\{\mu_0, \dots, \mu_r\}$ and $\{n_1, \dots, n_r\}$ depending on (x, x') (r may also depend on (x, x')) satisfies the inequality :*

$$1 + \left\lceil \sum_{i=0}^r \frac{\sqrt[n_i]{2} n_i}{\sqrt[n_i]{\mu_i}} \right\rceil \leq \left\lfloor \frac{C}{\sqrt[n]{\mu_0}} \right\rfloor.$$

3.4 Estimate of the degree

To relate μ_0 and the degree of the pluricanonical image of X , we need the following lemma.

Lemma 3.11 *If $\Phi_{|mK_X|}$ is birational rational map onto its image, then*

$$\deg \Phi_{|mK_X|}(X) \leq \mu_0 \cdot m^n$$

holds.

Proof. Let $p : \tilde{X} \rightarrow X$ be the resolution of the base locus of $|mK_X|$ and let

$$p^* |mK_X| = |P_m| + F_m$$

be the decomposition into the free part $|P_m|$ and the fixed component F_m . We have

$$\deg \Phi_{|mK_X|}(X) = P_m^n,$$

holds. Then by the ring structure of $R(X, K_X)$, we have an injection

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\nu P_m)) \rightarrow H^0(X, \mathcal{O}_X(m\nu K_X))$$

for every $\nu \geq 1$. We note that since $\mathcal{O}_{\tilde{X}}(\nu P_m)$ is globally generated on \tilde{X} , for every $\nu \geq 1$ we have the injection

$$\mathcal{O}_{\tilde{X}}(\nu P_m) \rightarrow p^* \mathcal{O}_X(m\nu K_X).$$

Hence there exists a natural morphism

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\nu P_m)) \rightarrow H^0(X, \mathcal{O}_X(m\nu K_X))$$

for every $\nu \geq 1$. This morphism is clearly injective. This implies that

$$\mu_0 \geq m^{-n} \mu(\tilde{X}_i, P_m)$$

holds. Since P_m is nef and big on X , we see that

$$\mu(\tilde{X}, P_m) = P_m^n$$

holds. Hence

$$\mu_0 \geq m^{-n} P_m^n$$

holds. This implies that

$$\deg \Phi_{|mK_X|}(X) \leq \mu_0 \cdot m^n$$

holds. **Q.E.D.**

3.5 Completion of the proof of Theorem 1.1 and 1.2 assuming MMP

By Lemma 3.9.3.10 and 3.11 we see that if $\mu_0 \leq 1$ holds, for

$$m := \lfloor \frac{C}{\sqrt[n]{\mu_0}} \rfloor,$$

$|mK_X|$ gives a birational embedding of X and

$$\deg \Phi_{|mK_X|}(X) \leq C^n$$

holds, where C is the positive constant in Lemma 3.10. Also

$$\dim H^0(X, \mathcal{O}_X(mK_X)) \leq n + 1 + \deg \Phi_{|mK_X|}(X)$$

holds by the semipositivity of the Δ -genus ([8]). Hence we have that if $\mu_0 \leq 1$,

$$\dim H^0(X, \mathcal{O}_X(mK_X)) \leq n + 1 + C^n$$

holds.

Since C is a positive constant depending only on n , combining the above two inequalities, we have that there exists a positive constant $C(n)$ depending only on n such that

$$\mu_0 = K_X^n \geq C(n)$$

holds.

More precisely we argue as follows. Let \mathcal{H} be an irreducible component of the Hilbert scheme of a projective spaces of dimension $\leq n + C^n$. Let \mathcal{H}_0 be the Zariski open subset of \mathcal{H} which parametrizes irreducible subvarieties. Then there exists a finite stratification of \mathcal{H}_0 by Zariski locally closed subsets such that on each stratum there exists a simultaneous resolution of the universal family on the strata. We note that the volume of the canonical bundle of the resolution is constant on each strata by [31, 22]. Hence there exists a positive constant $C(n)$ depending only on n such that

$$\mu(X, K_X) \geq C(n)$$

holds for every projective n -fold X of general type by the degree bound as above in the case of $\mu_0 \leq 1$. This completes the proof of Theorem 1.2 assuming MMP.

Then by Lemma 3.9 and 3.10, we see that there exists a positive integer ν_n depending only on n such that for every projective n -fold X of general type, $|mK_X|$ gives a birational embedding into a projective space for every $m \geq \nu_n$. This completes the proof of Theorem 1.1 assuming MMP.

4 Proof of Theorem 1.1 and 1.2 without assuming MMP

In this section we shall prove Theorem 1.1 and 1.2 in full generality. The proof is almost parallel to the one assuming MMP, if we replace the minimal model by an AZD (analytic Zariski decomposition) of the canonical line bundle.

4.1 Analytic Zariski decomposition

To study a pseudoeffective line bundle we introduce the notion of analytic Zariski decompositions. By using analytic Zariski decompositions, we can handle a pseudoeffective line bundle, as if it were a nef line bundle.

Definition 4.1 *Let M be a compact complex manifold and let L be a line bundle on M . A singular hermitian metric h on L is said to be an analytic Zariski decomposition, if the followings hold.*

1. Θ_h is a closed positive current,
2. for every $m \geq 0$, the natural inclusion

$$H^0(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h^m)) \rightarrow H^0(M, \mathcal{O}_M(mL))$$

is isomorphism.

Remark 4.1 *If an AZD exists on a line bundle L on a smooth projective variety M , L is pseudoeffective by the condition 1 above.*

Theorem 4.1 ([26, 27]) *Let L be a big line bundle on a smooth projective variety M . Then L has an AZD.*

As for the existence for general pseudoeffective line bundles, now we have the following theorem.

Theorem 4.2 ([6, Theorem 1.5]) *Let X be a smooth projective variety and let L be a pseudoeffective line bundle on X . Then L has an AZD.*

Although the proof is in [6], we shall give a proof here, because we shall use it afterward.

Let h_0 be a fixed C^∞ -hermitian metric on L . Let E be the set of singular hermitian metric on L defined by

$$E = \{h; h : \text{lowersemicontinuous singular hermitian metric on } L,$$

$$\Theta_h \text{ is positive, } \frac{h}{h_0} \geq 1\}.$$

Since L is pseudoeffective, E is nonempty. We set

$$h_L = h_0 \cdot \inf_{h \in E} \frac{h}{h_0},$$

where the infimum is taken pointwise. The supremum of a family of plurisubharmonic functions uniformly bounded from above is known to be again plurisubharmonic, if we modify the supremum on a set of measure 0 (i.e., if we take the uppersemicontinuous envelope) by the following theorem of P. Lelong.

Theorem 4.3 ([17, p.26, Theorem 5]) *Let $\{\varphi_t\}_{t \in T}$ be a family of plurisubharmonic functions on a domain Ω which is uniformly bounded from above on every compact subset of Ω . Then $\psi = \sup_{t \in T} \varphi_t$ has a minimum uppersemicontinuous majorant ψ^* which is plurisubharmonic. We call ψ^* the uppersemicontinuous envelope of ψ .*

Remark 4.2 *In the above theorem the equality $\psi = \psi^*$ holds outside of a set of measure 0 (cf. [17, p.29]).*

By Theorem 4.3, we see that h_L is also a singular hermitian metric on L with $\Theta_h \geq 0$. Suppose that there exists a nontrivial section $\sigma \in \Gamma(X, \mathcal{O}_X(mL))$ for some m (otherwise the second condition in Definition 4.1 is empty). We note that

$$\frac{1}{|\sigma|^{\frac{2}{m}}}$$

gives the weight of a singular hermitian metric on L with curvature $2\pi m^{-1}(\sigma)$, where (σ) is the current of integration along the zero set of σ . By the construction we see that there exists a positive constant c such that

$$(\star) \quad \frac{h_0}{|\sigma|^{\frac{2}{m}}} \geq c \cdot h_L$$

holds. Hence

$$\sigma \in H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_\infty(h_L^m))$$

holds. Hence in particular

$$\sigma \in H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h_L^m))$$

holds. This means that h_L is an AZD of L . **Q.E.D.**

Remark 4.3 *By the above proof (cf. (\star)) we have that for the AZD h_L constructed as above*

$$H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_\infty(h_L^m)) \simeq H^0(X, \mathcal{O}_X(mL))$$

holds for every m , where $\mathcal{I}_\infty(h_L^m)$ denotes the L^∞ -multiplier ideal sheaf, i.e., for every open subset U in X ,

$$\mathcal{I}_\infty(h_L^m)(U) := \{f \in \mathcal{O}_X(U) \mid |f|^2 (h_L/h_0)^m \in L_{loc}^\infty(U)\}.$$

4.2 The L^2 -extension theorem

Let M be a complex manifold of dimension n and let S be a closed complex submanifold of M . Then we consider a class of continuous function $\Psi : M \rightarrow [-\infty, 0)$ such that

1. $\Psi^{-1}(-\infty) \supset S$,
2. if S is k -dimensional around a point x , there exists a local coordinate (z_1, \dots, z_n) on a neighbourhood of x such that $z_{k+1} = \dots = z_n = 0$ on $S \cap U$ and

$$\sup_{U \setminus S} |\Psi(z) - (n-k) \log \sum_{j=k+1}^n |z_j|^2| < \infty.$$

The set of such functions Ψ will be denoted by $\sharp(S)$.

For each $\Psi \in \sharp(S)$, one can associate a positive measure $dV_M[\Psi]$ on S as the minimum element of the partially ordered set of positive measures $d\mu$ satisfying

$$\int_{S_k} f d\mu \geq \overline{\lim}_{t \rightarrow \infty} \frac{2(n-k)}{v_{2n-2k-1}} \int_M f \cdot e^{-\Psi} \cdot \chi_{R(\Psi, t)} dV_M$$

for any nonnegative continuous function f with $\text{supp } f \subset \subset M$. Here S_k denotes the k -dimensional component of S , v_m denotes the volume of the unit sphere in \mathbf{R}^{m+1} and $\chi_{R(\Psi, t)}$ denotes the characteristic function of the set

$$R(\Psi, t) = \{x \in M \mid -t - 1 < \Psi(x) < -t\}.$$

Let M be a complex manifold and let (E, h_E) be a holomorphic hermitian vector bundle over M . Given a positive measure $d\mu_M$ on M , we shall denote $A^2(M, E, h_E, d\mu_M)$ the space of L^2 holomorphic sections of E over M with respect to h_E and $d\mu_M$. Let S be a closed complex submanifold of M and let $d\mu_S$ be a positive measure on S . The measured submanifold $(S, d\mu_S)$ is said to be a set of interpolation for $(E, h_E, d\mu_M)$, or for the sapce $A^2(M, E, h_E, d\mu_M)$, if there exists a bounded linear operator

$$I : A^2(S, E \mid_S, h_E, d\mu_S) \longrightarrow A^2(M, E, h_E, d\mu_M)$$

such that $I(f) \mid_S = f$ for any f . I is called an interpolation operator. The following theorem is crucial.

Theorem 4.4 ([24, Theorem 4]) *Let M be a complex manifold with a continuous volume form dV_M , let E be a holomorphic vector bundle over M with C^∞ -fiber metric h_E , let S be a closed complex submanifold of M , let $\Psi \in \sharp(S)$ and let K_M be the canonical bundle of M . Then $(S, dV_M(\Psi))$ is a set of interpolation for $(E \otimes K_M, h_E \otimes (dV_M)^{-1}, dV_M)$, if the followings are satisfied.*

1. *There exists a closed set $X \subset M$ such that*
 - (a) *X is locally negligble with respect to L^2 -holomorphic functions, i.e., for any local coordinate neighbourhood $U \subset M$ and for any L^2 -holomorphic function f on $U \setminus X$, there exists a holomorphic function \tilde{f} on U such that $\tilde{f} \mid U \setminus X = f$.*
 - (b) *$M \setminus X$ is a Stein manifold which intersects with every component of S .*
2. $\Theta_{h_E} \geq 0$ *in the sense of Nakano,*
3. $\Psi \in \sharp(S) \cap C^\infty(M \setminus S)$,
4. $e^{-(1+\epsilon)\Psi} \cdot h_E$ *has semipositive curvature in the sense of Nakano for every $\epsilon \in [0, \delta]$ for some $\delta > 0$.*

Under these conditions, there exists a constant C and an interpolation operator from $A^2(S, E \otimes K_M \mid_S, h \otimes (dV_M)^{-1} \mid_S, dV_M[\Psi])$ to $A^2(M, E \otimes K_M, h \otimes (dV_M)^{-1}, dV_M)$ whose norm does not exceed $C\delta^{-3/2}$. If Ψ is plurisubharmonic, the interpolation operator can be chosen so that its norm is less than $2^4\pi^{1/2}$.

The above theorem can be generalized to the case that (E, h_E) is a singular hermitian line bundle with semipositive curvature current (we call such a singular hermitian line bundle (E, h_E) a **pseudoeffective singular hermitian line bundle**) as was remarked in [24].

Lemma 4.1 *Let $M, S, \Psi, dV_M, dV_M[\Psi], (E, h_E)$ be as in Theorem 4.2. Let (L, h_L) be a pseudoeffective singular hermitian line bundle on M . Then S is a set of interpolation for $(K_M \otimes E \otimes L, dV_M^{-1} \otimes h_E \otimes h_L)$.*

4.3 A construction of the function Ψ

Let M be a smooth projective n -fold and let S be a k -dimensional (not necessary smooth) subvariety of M . Let $\mathcal{U} = \{U_\gamma\}$ be a finite Stein covering of M and let $\{f_1^{(\gamma)}, \dots, f_{m(\gamma)}^{(\gamma)}\}$ be a gnerator of the ideal sheaf associated with S on U_γ . Let $\{\phi_\gamma\}$ be a partition of unity subordinates to \mathcal{U} . We set

$$\Psi := (n - k) \sum_{\gamma} \phi_\gamma \cdot \left(\sum_{\ell=1}^{m(\gamma)} |f_\ell^{(\gamma)}|^2 \right).$$

Then the residue volume form $dV[\Psi]$ is defined as in the last subsection. Here the residue volume form $dV[\Psi]$ of an continuous volume form dV on M is not well defined on the singular locus of S . But this is not a difficulty to apply Theorem 4.3 or Lemma 4.1, since there exists a proper Zariski closed subset Y of X such that $(X - Y) \cap S$ is smooth.

4.4 Volume of pseudoeffective line bundles

To measure the positivity of big line bundles on a projective variety, we shall introduce the notion of volume of a projective variety with respect to a big line bundle.

Definition 4.2 *Let L be a line bundle on a compact complex manifold M of dimension n . We define the L -volume of M by*

$$\mu(M, L) := n! \cdot \overline{\lim}_{m \rightarrow \infty} m^{-n} \dim H^0(M, \mathcal{O}_M(mL)).$$

With respect to a pseudoeffective singular hermitian line bundle (cf. for the definition of pseudoeffective singular hermitian line bundles the last part of 4.2), we define the volume as follows.

Definition 4.3 ([28]) *Let (L, h) be a pseudoeffective singular hermitian line bundle on a smooth projective variety X of dimension n . We define the volume $\mu(X, L)$ of X with respect to (L, h) by*

$$\mu(X, (L, h)) := n! \cdot \overline{\lim}_{m \rightarrow \infty} m^{-n} \dim H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^m)).$$

A pseudoeffective singular hermitian line bundle (L, h) is said to be big, if $\mu(X, (L, h)) > 0$ holds.

We may consider $\mu(X, (L, h))$ as the intersection number $(L, h)^n$. Let Y be a subvariety of X of dimension d and let $\pi_Y : \tilde{Y} \rightarrow Y$ be a resolution of Y . We define $\mu(Y, (L, h) |_Y)$ as

$$\mu(Y, (L, h) |_Y) := \mu(\tilde{Y}, \pi_Y^*(L, h)).$$

The righthandside is independent of the choice of the resolution π because of the remark below.

Remark 4.4 In Definition 4.3, let $\pi : \tilde{X} \rightarrow X$ be any modification. Then

$$\mu(X, (L, h)) = \mu(\tilde{X}, \pi^*(L, h))$$

holds, since

$$\pi_*(\mathcal{O}_{\tilde{X}}(K_{\tilde{X}}) \otimes \mathcal{I}(\pi^*h^m)) = \mathcal{O}_X(K_X) \otimes \mathcal{I}(h^m)$$

holds for every m and

$$\overline{\lim}_{m \rightarrow \infty} m^{-n} \dim H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^m)) = \overline{\lim}_{m \rightarrow \infty} m^{-n} \dim H^0(X, \mathcal{O}_X(mL+D) \otimes \mathcal{I}(h^m))$$

holds for any Cartier divisor D on X . This last equality can be easily checked, if D is a smooth irreducible divisor, by using the exact sequence

$$0 \rightarrow \mathcal{O}_X(mL) \otimes \mathcal{I}(h^m) \rightarrow \mathcal{O}_X(mL+D) \otimes \mathcal{I}(h^m) \rightarrow \mathcal{O}_D(mL+D) \otimes \mathcal{I}(h^m) \rightarrow 0.$$

For a general D , the equality follows by expressing D as a difference of two very ample divisors.

4.5 Construction of stratifications

Let X be a smooth projective n -fold of general type. Let h be an AZD of K_X constructed as in Section 4.1. We may assume that h is lowersemicontinuous (cf. [27, 6]). This is a technical assumption so that a local potential of the curvature current of h is plurisubharmonic. This is used to restrict h to a subvariety of X (if we only assume that the local potential is only locally integrable, the restriction is not well defined). We set

$$X^\circ = \{x \in X \mid x \notin \text{Bs} \mid mK_X \mid \text{ and } \Phi_{|mK_X|} \text{ is a biholomorphism}$$

$$\text{ on a neighbourhood of } x \text{ for some } m \geq 1\}$$

as before. We set

$$\mu_0 := \mu(X, (K_X, h)) = \mu(X, K_X).$$

The last equality holds, since h is an AZD of K_X . We note that for every $x \in X^\circ$, $\mathcal{I}(h^m)_x \simeq \mathcal{O}_{X,x}$ holds for every $m \geq 0$ (cf. [27] or [6, Theorem 1.5]). Using this fact the proof of the following lemma is identical to that of Lemma 4.1.

Lemma 4.2 Let x, x' be distinct points on X° . We set

$$\mathcal{M}_{x,x'} = \mathcal{M}_x \otimes \mathcal{M}_{x'},$$

where $\mathcal{M}_x, \mathcal{M}_{x'}$ denote the maximal ideal sheaf of the points x, x' respectively. Let ε be a sufficiently small positive number. Then

$$H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{M}_{x,x'}^{\lceil \sqrt[m]{\mu_0}(1-\varepsilon) \frac{m}{\sqrt{2}}} \rceil}) \neq 0$$

for every sufficiently large m .

Let us take a sufficiently large positive integer m_0 and let σ_0 be a general (nonzero) element of $H^0(X, \mathcal{O}_X(m_0 K_X) \otimes \mathcal{M}_{x, x'}^{\lceil \sqrt[m_0]{\mu_0}(1-\varepsilon) \frac{m_0}{\sqrt{2}}} \rceil})$. We set

$$D_0 := \frac{1}{m_0}(\sigma_0)$$

and

$$h_0 = \frac{1}{|\sigma_0|^{2/m_0}}.$$

Let us define a positive (rational) number α_0 by

$$\alpha_0 := \inf\{\alpha > 0 \mid \text{Supp}(\mathcal{O}_X/\mathcal{I}(h_0^\alpha)) \text{ contains both } x \text{ and } x'\}.$$

This is essentially the same definition as in the last section. Then as before we see that

$$\alpha_0 \leq \frac{n \sqrt[n]{2}}{\sqrt[n]{\mu_0}(1-\varepsilon)}$$

holds. Suppose that for every small positive number δ , $\text{Supp}(\mathcal{O}_X/\mathcal{I}(h_0^{\alpha-\delta}))$ does not contain both x and x' . Other cases will be treated as before. Let X_1 be the minimal center of log canonical singularities at x . We define the positive number μ_1 by

$$\mu_1 := \mu(X_1, (K_X, h) |_{X_1}).$$

Then since $x, x' \in X^\circ$, μ_1 is positive.

For the later purpose, we shall perturb h_0 so that X_1 is the only center of log canonical singularities at x . Let E be an effective \mathbf{Q} -divisor such that $K_X - E$ is ample. By the definition of X° , we may assume that the support of E does not contain x . Let a be a positive integer such that $A := a(K_X - E)$ is a very ample Cartier divisor such that $\mathcal{O}_X(A) \otimes \mathcal{I}_{X_1}$ is globally generated. Let $\rho_1, \dots, \rho_e \in H^0(X, \mathcal{O}_X(A) \otimes \mathcal{I}_{X_1})$ be a set of generators of $\mathcal{O}_X(A) \otimes \mathcal{I}_{X_1}$ on X . Then if we replace h_0 by

$$\frac{1}{(|\sigma_0|^2 (\sum_{i=1}^e |\rho_i|^2))^{\frac{1}{m_0+a}}}$$

has the desired property. If we take m_0 very large (in comparison with a), we can make the new α_0 arbitrary close to the original α_0 . To proceed further we use essentially the same procedure as in the previous section. The only essential difference here is to use Lemma 4.1 instead of Lemma 3.3. Let x_1, x_2 be two distinct regular points on $X_1 \cap X^\circ$. Let m_1 be a sufficiently large positive integer and Let

$$\sigma'_1 \in H^0(X_1, \mathcal{O}_{X_1}(m_1 K_X) \otimes \mathcal{I}(h^{m_1}) \cdot \mathcal{M}_{x_1, x_2}^{\lceil \sqrt[m_1]{\mu_1}(1-\varepsilon) \frac{m_1}{\sqrt{2}}} \rceil})$$

be a nonzero element.

By Kodaira's lemma there is an effective \mathbf{Q} -divisor E such that $K_X - E$ is ample. By the definition of X° , we may assume that the support of E does not contain both x and x' . Let ℓ_1 be a sufficiently large positive integer which will be specified later such that

$$L_1 := \ell_1(K_X - E)$$

is Cartier. Let h_{L_1} be a C^∞ -hermitian metric on L_1 with strictly positive curvature. Let τ be a nonzero section in $H^0(X, \mathcal{O}_X(L_1))$. We set

$$\Psi := \alpha_0 \log \frac{h_0}{h}.$$

Let dV be a C^∞ -volume form on X . We note that the residue volume form $dV[\Psi]$ on X_1 may have pole along some proper subvarieties in X_1 . By taking ℓ_1 sufficiently large and taking τ properly, we may assume that $h_{L_1}(\tau, \tau) \cdot dV[\Psi]$ is nonsingular on X_1 in the sense that the pullback of it to a nonsingular model of X_1 is nonsingular. Then by applying Lemma 4.1 for (E, h_E) to be $(([1 + \alpha_0]K_X, h^{[1+\alpha_0]})$ and $(X, X_1, \Psi, dV, dV[\Psi], ((m_1 - [\alpha_0] - 2)K_X + L_1, h^{m_1-1}h_{L_1}))$ we see that

$$\sigma'_1 \otimes \tau \in H^0(X_1, \mathcal{O}_{X_1}(m_1 K_X + L_1) \otimes \mathcal{I}(h^{m_1}) \cdot \mathcal{M}_{x_1, x_2}^{\lceil \frac{n\sqrt{\mu_1}(1-\varepsilon)}{n\sqrt{2}} \rceil})$$

extends to a section

$$\sigma_1 \in H^0(X, \mathcal{O}_X((m_1 + \ell_1)K_X)).$$

We note that even though $dV[\Psi]$ may have singularity on X_1 , we may apply Lemma 4.1, because there exists a proper Zariski closed subset Y of X such that the restriction of $dV[\Psi]$ to $(X - Y) \cap X_1$ is smooth. Of course the singularity of $dV[\Psi]$ affects to the L^2 -condition. But this has already been handled by the boundedness of $h_{L_1}(\tau, \tau) \cdot dV[\Psi]$.

Then by entirely the same procedure as in Section 3, for distinct points x, x' , we construct a strictly decreasing sequence of subvarieties

$$X = X_0 \supset X_1 \supset \cdots \supset X_r \supset X_{r+1} = x \text{ or } x'$$

and invariants (depending on small positive numbers $\varepsilon_0, \dots, \varepsilon_{r-1}$, large positive integers m_0, m_1, \dots, m_r , etc.) :

$$\alpha_0, \alpha_1, \dots, \alpha_r,$$

$$\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{r-1}$$

$$\mu_0, \mu_1, \dots, \mu_r$$

and

$$n > n_1 > \cdots > n_r$$

inductively. By Nadel's vanishing theorem ([21, p.561]) we have the following lemma.

Lemma 4.3 *Let x, x' be two distinct points on X° . Then for every $m \geq [\sum_{i=0}^r \alpha_i] + 1$, $\Phi_{|mK_X|}$ separates x and x' .*

$$\alpha_i \leq \frac{n_i \sqrt[n_i]{2}}{\sqrt[n_i]{\mu_i}} + \delta$$

hold for $1 \leq i \leq r$.

We may also construct the stratification as a family as in Section 3.2.

4.6 Another subadjunction theorem

Let M be a smooth projective variety and let (L, h_L) be a singular hermitian line bundle on M such that $\Theta_{h_L} \geq 0$ on M . Let dV be a C^∞ -volume form on M . Let $\sigma \in \Gamma(M, \mathcal{O}_M(m_0 L) \otimes \mathcal{I}(h))$ be a global section. Let α be a positive rational number ≤ 1 and let S be an irreducible subvariety of M such that $(M, \alpha(\sigma))$ is log canonical but not KLT (Kawamata log-terminal) on the generic point of S and $(M, (\alpha - \epsilon)(\sigma))$ is KLT on the generic point of S for every $0 < \epsilon \ll 1$. We set

$$\Psi = \alpha \log h_L(\sigma, \sigma).$$

Suppose that S is smooth (if S is not smooth, we just need to take an embedded resolution and apply Theorem 4.4 below). We shall assume that S is not contained in the singular locus of h , where the singular locus of h means the set of points where h is $+\infty$. Then as in Section 4.2, we may define a (possibly singular measure) $dV[\Psi]$ on S . This can be viewed as follows. Let $f : N \rightarrow M$ be a log-resolution of $(X, \alpha(\sigma))$. Then as before we may define the singular volume form $f^* dV[f^* \Psi]$ on the divisorial component of $f^{-1}(S)$. The singular volume form $dV[\Psi]$ is defined as the fibre integral of $f^* dV[f^* \Psi]$. Let $d\mu_S$ be a C^∞ -volume form on S and let φ be the function on S defined by

$$\varphi := \log \frac{dV[\Psi]}{d\mu_S}$$

($dV[\Psi]$ may be singular on a subvariety of S , also it may be totally singular on S).

Theorem 4.5 ([32, Theorem 5.1]) *Let M, S, Ψ be as above. Suppose that S is smooth. Let d be a positive integer such that $d \geq \alpha m_0$. Then every element of $A^2(S, \mathcal{O}_S(m(K_M + dL)), e^{-(m-1)\varphi} \cdot dV^{-m} \otimes h_L^m|_S, dV[\Psi])$ extends to an element of*

$$H^0(M, \mathcal{O}_M(m(K_M + dL))).$$

As we mentioned as above the smoothness assumption on S is just to make the statement simpler.

Theorem 4.5 follows from Theorem 4.6 below by minor modifications (cf. [32]). The main difference is the fact that the residue volume form $dV[\Psi]$ is singular on S . But this does not affect the proof, since in the L^2 -extension theorem (Theorem 4.4) we do not need to assume that the manifold M is compact. Hence we may remove a suitable subvarieties so that we do not need to consider the pole of $dV[\Psi]$ on S (but of course the pole of $dV[\Psi]$ affects the L^2 -conditions).

Theorem 4.6 *Let M be a projective manifold with a continuous volume form dV , let L be a holomorphic line bundle over M with a C^∞ -hermitian metric h_L , let S be a compact complex submanifold of M , let $\Psi : M \rightarrow [-\infty, 0)$ be a continuous function and let K_M be the canonical bundle of M .*

1. $\Psi \in \sharp(S) \cap C^\infty(M \setminus S)$ (As for the definition of $\sharp(S)$, see Section 4.2),
2. $\Theta_{h \cdot e^{-(1+\epsilon)\Psi}} \geq 0$ for every $\epsilon \in [0, \delta]$ for some $\delta > 0$,
3. there is a positive line bundle on M .

Then every element of $H^0(S, \mathcal{O}_S(m(K_M+L)))$ extends to an element of $H^0(M, \mathcal{O}_M(m(K_M+L)))$.

For the completeness we shall give a proof of Theorem 4.5 under the additional condition :

Condition $(K_M + L, dV^{-1} \otimes h_L)$ is big (cf. Definition 4.3).

The reason why we put this condition is that we only need Theorem 4.5 and 4.6 under this condition.

Let M, S, L be as in Theorem 4.6. Let n be the dimension of M . Let h_S be a canonical AZD ([27]) of $K_M + L|_S$. Let A be a sufficiently ample line bundle on M . We consider the Bergman kernel

$$K(S, A + m(K_M + L)|_S, h_A \cdot h_S^{m-1} \cdot dV^{-1} \cdot h_L, d\Psi_S) = \sum_i |\sigma_i^{(m)}|^2,$$

where $\{\sigma_i^{(m)}\}$ is a complete orthonormal basis of $A^2(S, A + m(K_M + L)|_S, h_A \cdot h_S^{m-1} \cdot dV^{-1} \cdot h_L, d\Psi_S)$. We note that (cf. [16, p.46, Proposition 1.4.16])

$$\begin{aligned} & K(S, A + m(K_M + L)|_S, h_A \cdot h_S^{m-1} \cdot dV^{-1} \cdot h_L, d\Psi_S)(x) \\ &= \sup\{|\sigma|^2(x) \mid \sigma \in A^2(S, A + m(K_M + L)|_S, h_A \cdot h_S^{m-1} \cdot dV^{-1} \cdot h_L, d\Psi_S), \|\sigma\| = 1\} \end{aligned}$$

holds for every $x \in S$.

Let us define the singular hermitian metric on $m(K_M + L)|_S$ by

$$h_{m,S} := K(A + m(K_M + L)|_S, h_A \cdot h_S^{m-1} \cdot dV^{-1} \cdot h_L, d\Psi_S)^{-1}$$

Then as in [5, Section 4] and [27], we see that

$$h_S := \liminf_{m \rightarrow \infty} \sqrt[m]{h_{m,S}}$$

holds. Hence $\{\sqrt[m]{h_{m,S}}\}$ is considered to be an algebraic approximation of h_S . We note that there exists a positive constant C_0 independent of m such that

$$h_{m,S} \leq C_0 \cdot h_A \cdot h_S^m$$

holds for every $m \geq 1$ as in [27]. Let h_M be a canonical AZD of $K_M + L$. By the assumption, $(K_M + L, h_M)$ is big, i.e.,

$$\lim_{m \rightarrow \infty} \frac{\log \dim H^0(M, \mathcal{O}_M(A + m(K_M + L)) \otimes \mathcal{I}(h_M^m))}{\log m} = n.$$

Let $\nu(S)$ denote the numerical Kodaira dimension of $(K_M + L|_S, h_S)$ (for our purpose we may assume that $\nu(S) = \dim S$), i.e.,

$$\nu(S) := \lim_{m \rightarrow \infty} \frac{\log \dim H^0(S, \mathcal{O}_S(A + m(K_M + L)) \otimes \mathcal{I}(h_S^m))}{\log m}.$$

Inductively on m , we extend each

$$\sigma \in A^2(S, A + m(K_M + L)|_S, h_A \cdot h_S^{m-1} \cdot dV^{-1} \cdot h_L, d\Psi_S)$$

to a section

$$\tilde{\sigma} \in A^2(M, A + m(K_M + L), dV^{-1} \cdot h_L \cdot \tilde{h}_{m-1}, dV)$$

with the estimate

$$\|\tilde{\sigma}\| \leq C \cdot m^{-(n-\nu(S))} \|\sigma\|,$$

where $\|\cdot\|$'s denote the L^2 -norms respectively, C is a positive constant independent of m and we have defined

$$\tilde{K}_m(x) := \sup\{|\tilde{\sigma}|^2(x) \mid \|\tilde{\sigma}\|_S = 1, \|\tilde{\sigma}\| \leq C \cdot m^{-(n-\nu(S))}\}$$

and set

$$\tilde{h}_m = \frac{1}{\tilde{K}_m}.$$

If we take C sufficiently large, then \tilde{h}_m is well defined for every $m \geq 0$. Here we note that the factor $m^{-(n-\nu(S))}$ comes from that h_M is dominated by a singular hermitian metric with strictly positive curvature by Kodaira's lemma by the condition (cf. [25, p.105,(1,11)]). By easy inductive estimates, we see that

$$\tilde{h}_\infty := \liminf_{m \rightarrow \infty} \sqrt[m]{\tilde{h}_m}$$

exists and gives an extension of h_S . In fact $h_A^{1/m}(\tilde{K}_m)^{1/m}$ is uniformly bounded from above as in [31, p.127, Lemma 3.3]. Then by [24] for every $m \geq 1$, we may extend every element of $A^2(m(K_M + L) |_S, dV^{-1} \cdot h_L \cdot h_S^{m-1}, dV[\Psi])$ to $A^2(m(K_M + L), dV^{-1} \cdot h_L \cdot \tilde{h}_\infty, dV)$. This completes the proof of Theorem 4.6.

4.7 Positivity result

The following positivity theorem is a key to the proof of Theorem 1.1 and 1.2.

Theorem 4.7 ([11, Theorem 2]) *Let $f : X \rightarrow B$ be a surjective morphism of smooth projective varieties with connected fibers. Let $P = \sum P_j$ and $Q = \sum_\ell Q_\ell$ be normal crossing divisors on X and B respectively, such that $f^{-1}(Q) \subset P$ and f is smooth over $B \setminus Q$. Let $D = \sum d_j P_j$ be a \mathbf{Q} -divisor on X , where d_j may be positive, zero or negative, which satisfies the following conditions :*

1. $D = D^h + D^v$ such that $f : \text{Supp}(D^h) \rightarrow B$ is surjective and smooth over $B \setminus Q$, and $f(\text{Supp}(D^h)) \subset Q$. An irreducible component of D^h (resp. D^v) is called horizontal (resp. vertical).
2. $d_j < 1$ for all j .
3. The natural homomorphism $\mathcal{O}_B \rightarrow f_* \mathcal{O}_X([-D])$ is surjective at the generic point of B .

4. $K_X + D \sim_{\mathbf{Q}} f^*(K_B + L)$ for some \mathbf{Q} -divisor L on B . Let

$$\begin{aligned} f^*Q_\ell &= \sum_j w_{\ell j} P_j \\ \bar{d}_j &:= \frac{d_j + w_{\ell j} - 1}{w_{\ell j}} \quad \text{if } f(P_j) = Q_\ell \\ \delta_\ell &:= \max\{\bar{d}_j; f(P_j) = Q_\ell\} \\ \Delta &:= \sum_\ell \delta_\ell Q_\ell \\ M &:= L - \Delta. \end{aligned}$$

Then M is nef.

Here the meaning of the divisor Δ may be difficult to understand. I would like to give an geometric interpretation of Δ . Let X, P, Q, D, B, Δ be as above. Let dV be a C^∞ -volume form on X . Let σ_j be a global section of $\mathcal{O}_X(P_j)$ with divisor P_j . Let $\|\sigma_j\|$ denote the hermitian norm of σ_j with respect to a C^∞ -hermitian metric on $\mathcal{O}_X(P_j)$ respectively. Let us consider the singular volume form

$$\Omega := \frac{dV}{\prod_j \|\sigma_j\|^{2d_j}}.$$

Then by taking the fiber integral of Ω with respect to $f : X \rightarrow B$, we obtain a singular volume form $\int_{X/B} \Omega$ on B . Then the divisor Δ corresponds to the singularity of the singular volume form $\int_{X/B} \Omega$ on B .

4.8 Use of two subadjunction theorems

This subsection is the counterpart of Section 3.3.

Lemma 4.4 *For every X_j ,*

$$\mu(W_j, K_{W_j}) \leq (\lceil (1 + \sum_{i=0}^{j-1} \alpha_i) \rceil)^{n_j} \cdot \mu_j$$

holds, where W_j is a nonsingular model of X_j (we note that $\mu(W_j, K_{W_j})$ is independent of the choice of the nonsingular model W_j).

Proof.

Let us set

$$\beta_j := \varepsilon_{j-1} + \sum_{i=0}^{j-1} (\alpha_i - \varepsilon_i).$$

Let D_i denotes the divisor $m_i^{-1}(\sigma_i)$ and we set

$$D := \sum_{i=1}^{j-1} (\alpha_i - \varepsilon_i) D_i + \varepsilon_{j-1} D_{j-1}.$$

Let $\pi : Y \rightarrow X$ be a log resolution of (X, D) which factors through the embedded resolution $\varpi : W_j \rightarrow X_j$ of X_j . By the perturbation as in Section

4.5, we may assume that there exists a unique irreducible component F_j of the exceptional divisor with discrepancy -1 which dominates X_j . Let

$$\pi_j : F_j \longrightarrow W_j$$

be the natural morphism induced by the construction. We set

$$\pi^*(K_X + D) |_{F_j} = K_{F_j} + G.$$

We may assume that the support of G is a divisor with normal crossings. Then all the coefficients of the horizontal component G^h with respect to π_j are less than 1 because F_j is the unique exceptional divisor with discrepancy -1 .

Let dV be a C^∞ -volume form on the X . Let Ψ be the function defined by

$$\Psi := \log(h^{\beta_j} \cdot | \sigma_{j-1} |^{\frac{2\varepsilon_{j-1}}{m_{j-1}}} \cdot \prod_{i=0}^{j-1} | \sigma_i |^{\frac{2(\alpha_i - \varepsilon_i)}{m_i}}).$$

Then the residue $\text{Res}_{F_j}(\pi^*(e^{-\Psi} \cdot dV))$ of $\pi^*(e^{-\Psi} \cdot dV)$ to F_j is a singular volume form with algebraic singularities corresponding to the divisor G . Since every coefficient of G^h is less than 1, there exists a Zariski open subset W_j^0 of W_j such that $\text{Res}_{F_j}(\pi^*(e^{-\Psi} \cdot dV))$ is integrable on $\pi_j^{-1}(W_j^0)$.

Then the pullback of the residue $dV[\Psi]$ of $e^{-\Psi} \cdot dV$ (to X_j) to W_j is given by the fiber integral of the above singular volume form $\text{Res}_{F_j}(\pi^*(e^{-\Psi} \cdot dV))$ on F_j , i.e.,

$$\varpi^* dV[\Psi] = \int_{F_j/W_j} \text{Res}_{F_j}(\pi^*(e^{-\Psi} \cdot dV))$$

holds. By Theorem 4.7, we see that $(K_{F_j} + G) - \pi_j^*(K_{W_j} + \Delta)$ is nef, where Δ is the divisor defined as in Theorem 4.7. And as in [11, cf. Proof of Theorem 1], $\varpi_* \Delta$ is effective on X_j . We may assume that $\pi : Y \longrightarrow X$ is a simultaneous resolution of $\text{Bs} | m_i K_X |$ ($0 \leq i \leq j-1$). Let us decompose D_i ($0 \leq i \leq j-1$) as

$$\pi^* D_i = P_i + N_i,$$

where P_i is the free part and N_i is the fixed component. Let us consider the contribution of

$$\sum_{i=0}^{j-1} \frac{\alpha_i - \varepsilon_i}{m_i} N_i + \frac{\varepsilon_{j-1}}{m_{j-1}} N_{j-1}$$

to the divisor Δ . Let σ_Δ be a multivalued meromorphic section of the \mathbf{Q} -line bundle $\mathcal{O}_{W_j}(\Delta)$ with divisor Δ . Let h_Δ be a C^∞ -hermitian metric on the \mathbf{Q} -line bundle $\mathcal{O}_{W_j}(\Delta)$. Then since

$$\sigma_i \in H^0(X, \mathcal{O}_X(m_i K_X) \otimes \mathcal{I}_\infty(h^{m_i}))$$

for every $0 \leq i \leq j-1$ (cf. Remark 4.3), we see that

$$h_\Delta(\sigma_\Delta, \sigma_\Delta) = O(\varpi^*((dV \cdot h^{-1}) |_{X_j})^{\beta_j})$$

holds. Hence we see that

$$\mu(W_j, K_{W_j}) \leq \mu(X_j, (([1 + \beta_j])K_X, dV^{-1} \otimes h^{[\beta_j]}) |_{X_j}) \quad (1)$$

holds.

Let us recall the interpretation of the divisor Δ in Section 4.7. Let dV_{W_j} be a C^∞ -volume form on W_j . Then there exists a positive constant $C > 1$ such that

$$\varpi^* dV[\Psi] = \int_{F_j/W_j} \text{Res}_{F_j}(\pi^*(e^{-\Psi} \cdot dV)) \leq C \cdot \frac{\varpi^*(dV \cdot h^{-1})^{\beta_j}}{h_\Delta(\sigma_\Delta, \sigma_\Delta)} \cdot dV_W$$

hold. We note that $dV \cdot h^{-1}$ is bounded from above by the construction of h ,

Since $\varpi_* \Delta$ is effective on X_j , by applying Theorem 4.5, we have the interpolation :

$$A^2(W_j, m(\lceil 1 + \beta_j \rceil)K_X, e^{-(m-1)\varphi} \cdot dV^{-m} \otimes h^{m\lceil \beta_j \rceil}, dV[\Psi]) \rightarrow H^0(X, \mathcal{O}_X(m(\lceil 1 + \beta_j \rceil)K_X)),$$

where φ is the weight function defined as in Theorem 4.5.

Since h is an AZD of K_X constructed as in Section 4.1, we see that every element of $H^0(X, \mathcal{O}_X(m(\lceil 1 + \beta_j \rceil)K_X))$ is bounded on X with respect to $h^{m(\lceil 1 + \beta_j \rceil)}$ (cf. Remark 4.3). In particular the restriction of an element of $H^0(X, \mathcal{O}_X(m(\lceil 1 + \beta_j \rceil)K_X))$ to X_j is bounded with respect to $h^{\lceil 1 + \beta_j \rceil}|_{X_j}$. Hence by the existence of the above extension, we have that

$$\mu(W_j, K_{W_j}) \leq \mu(X_j, (\lceil 1 + \beta_j \rceil)K_X, h^{\lceil 1 + \beta_j \rceil}|_{X_j}) \quad (2)$$

holds. The difference between the inequalities (1) and (2) is that in (2) $dV^{-1} \otimes h^{\lceil \beta_j \rceil}|_{X_j}$ is replaced by $h^{\lceil 1 + \beta_j \rceil}|_{X_j}$. This is the only point where Theorem 4.5 is used.

By the trivial inequality

$$\beta_j \leq \sum_{i=0}^j \alpha_i.$$

we have that

$$\mu(W_j, K_{W_j}) \leq (\lceil 1 + \sum_{i=0}^{j-1} \alpha_i \rceil)^{n_j} (K_X, h)^{n_j} \cdot X_j$$

holds by the definition of $(K_X, h)^{n_j} \cdot X_j$. This is the desired inequality, since $\mu_j = (K_X, h)^{n_j} \cdot X_j$ holds by definition. **Q.E.D.**

4.9 Estimate of the degree

Lemma 4.5 *If $\Phi_{|mK_X|}$ is a birational rational map onto its image, then*

$$\deg \Phi_{|mK_X|}(X) \leq \mu_0 \cdot m^n$$

holds.

Proof. Let $p : \tilde{X} \rightarrow X$ be the resolution of the base locus of $|mK_X|$ and let

$$p^* |mK_X| = |P_m| + F_m$$

be the decomposition into the free part $|P_m|$ and the fixed component F_m . We have

$$\deg \Phi_{|mK_X|}(X) = P_m^n$$

holds. Then by the ring structure of $R(X, K_X)$, we have an injection

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\nu P_m)) \rightarrow H^0(X, \mathcal{O}_X(m\nu K_X) \otimes \mathcal{I}(h^{m\nu}))$$

for every $\nu \geq 1$, since the righthandside is isomorphic to $H^0(X, \mathcal{O}_X(m\nu K_X))$ by the definition of an AZD. We note that since $\mathcal{O}_{\tilde{X}}(\nu P_m)$ is globally generated on \tilde{X} , for every $\nu \geq 1$ we have the injection

$$\mathcal{O}_{\tilde{X}}(\nu P_m) \rightarrow p^*(\mathcal{O}_X(m\nu K_X) \otimes \mathcal{I}(h^{m\nu})).$$

Hence there exists a natural homomorphism

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\nu P_m)) \rightarrow H^0(X, \mathcal{O}_X(m\nu K_X) \otimes \mathcal{I}(h^{m\nu}))$$

for every $\nu \geq 1$. This homomorphism is clearly injective. This implies that

$$\mu_0 \geq m^{-n} \mu(\tilde{X}, P_m)$$

holds by the definition of μ_0 . Since P_m is nef and big on X , we see that

$$\mu(\tilde{X}, P_m) = P_m^n$$

holds. Hence

$$\mu_0 \geq m^{-n} P_m^n$$

holds. This implies that

$$\deg \Phi_{|mK_X|}(X) \leq \mu_0 \cdot m^n$$

holds. **Q.E.D.**

4.10 Completion of the proof of Theorem 1.1 and 1.2.

We shall complete the proofs of Theorem 1.1 and 1.2. Suppose that Theorem 1.2 holds for every projective varieties of general type of dimension $< n$, i.e., there exists a positive constants $\{C(k)(k < n)\}$ such that for every every smooth projective k -fold Y of general type

$$\mu(Y, K_Y) \geq C(k)$$

holds. Let X be a smooth projective variety of general type of dimension n . Let U_0 be a nonempty Zariski open subset of X with respect to **countable Zariski topology** such that for every $x \in U_0$ there exist no subvarieties of nongeneral type containing x . Such U_0 surely exists, since there exists no dominant family of subvarieties of nongeneral type in X (see Section 3.3). Then if $(x, x') \in U_0 \times U_0 - \Delta_X$, the stratum X_j as in Section 3.1 is of general type as before. By Lemma 4.4, we see that

$$C(n_j) \leq ([1 + \sum_{i=0}^{j-1} \alpha_i])^{n_j} \cdot \mu_j$$

holds for W_j . As in Lemma 3.10 in Section 3.3, we obtain the following lemma.

Lemma 4.6 *Suppose that $\mu_0 \leq 1$ holds. Then there exists a positive constant C depending only on n such that for every $(x, x') \in U_0 \times U_0 - \Delta_X$ the corresponding invariants $\{\mu_0, \dots, \mu_r\}$ and $\{n_1, \dots, n_r\}$ depending on (x, x') (r may also depend on (x, x')) satisfies the inequality :*

$$1 + \left\lceil \sum_{i=0}^r \frac{\sqrt[n_i]{2} n_i}{\sqrt[n_i]{\mu_i}} \right\rceil \leq \left\lfloor \frac{C}{\sqrt[n]{\mu_0}} \right\rfloor.$$

We note that the finite stratification of $U_0 \times U_0 - \Delta_X$ is unnecessary for the proof, because $\{n_1, \dots, n_r\}$ is a strictly decreasing sequence and this sequence has only finitely many possibilities. By Lemma 4.3 we see that for

$$m := \left\lfloor \frac{C}{\sqrt[n]{\mu_0}} \right\rfloor,$$

$|mK_X|$ separates points in U_0 . Hence $|mK_X|$ gives a birational embedding of X .

Then by Lemma 4.5, if $\mu_0 \leq 1$ holds,

$$\deg \Phi_{|mK_X|}(X) \leq C^n$$

holds. Also

$$\dim H^0(X, \mathcal{O}_X(mK_X)) \leq n + 1 + \deg \Phi_{|mK_X|}(X)$$

holds by the semipositivity of the Δ -genus ([8]). Hence we have that if $\mu_0 \leq 1$,

$$\dim H^0(X, \mathcal{O}_X(mK_X)) \leq n + 1 + C^n$$

holds.

Since C is a positive constant depending only on n , combining the above two inequalities, by the argument as in Section 3.5, we have that there exists a positive constant $C(n)$ depending only on n such that

$$\mu_0 \geq C(n)$$

holds.

Then as in Section 3.5 we see that there exists a positive integer ν_n depending only on n such that for every projective n -fold X of general type, $|mK_X|$ gives a birational embedding into a projective space for every $m \geq \nu_n$. This completes the proof of Theorem 1.1.

5 The Severi-Iitaka conjecture

Let X be a smooth projective variety. We set

$$Sev(X) := \{(f, [Y]) \mid f : X \longrightarrow Y \text{ dominant rational map and } Y \text{ is of general type}\},$$

where $[Y]$ denotes the birational class of Y . By Theorem 1.1 and [19, p.117, Proposition 6.5] we obtain the following theorem.

Theorem 5.1 *$Sev(X)$ is finite.*

Remark 5.1 *In the case of $\dim Y = 1$, Theorem 5.1 is known as Severi's theorem. In the case of $\dim Y = 2$, Theorem 5.1 has already been known by K. Maehara ([19]). In the case of $\dim Y = 3$, Theorem 5.1 has recently proved by T. Bandman and G. Dethloff ([2]).*

6 Appendix

6.1 Volume of nef and big line bundles

The following fact seems to be well known. But for the completeness, I would like to include the proof.

Proposition 6.1 *Let M be a smooth projective n -fold and let L be a nef and big line bundle on M . Then*

$$n! \cdot \overline{\lim}_{m \rightarrow \infty} m^{-n} \dim H^0(M, \mathcal{O}_M(mL)) = L^n$$

holds.

Proof of Proposition 5.1. Since L is big, there exists an effective \mathbf{Q} -divisor F such that $L - F$ is ample. Let a be a positive integer such that $A := a(L - F)$ is a very ample Cartier divisor and $A - K_X$ is ample. Then by the Kodaira vanishing theorem, for every $q \geq 1$,

$$H^q(M, \mathcal{O}_M(A + mL)) = 0$$

holds for every $m \geq 0$. By the Riemann-Roch theorem we have that

$$n! \cdot \overline{\lim}_{m \rightarrow \infty} m^{-n} \dim H^0(M, \mathcal{O}_M(A + mL)) = L^n$$

holds. By the definition of A , we see that

$$n! \cdot \overline{\lim}_{m \rightarrow \infty} m^{-n} \dim H^0(M, \mathcal{O}_M(mL)) = L^n$$

holds. This completes the proof. **Q.E.D.**

6.2 A Serre type vanishing theorem

Lemma 6.1 *Let X be a projective variety with only canonical singularities (cf. [15, p.56, Definition 2.34]). Let E be a vector bundle on X and let L be a nef line bundle on X . Let A be an ample line bundle on X . Then there exists a positive integer k_0 depending only on E such that for every $k \geq k_0$*

$$H^q(X, \mathcal{O}_X(K_X + mL + kA) \otimes E) = 0$$

holds for every $m \geq 0$ and $q \geq 1$.

Proof. Let ω_X be the L^2 -dualizing sheaf of X , i.e., the direct image sheaf of the canonical sheaf of a resolution of X . Since X has only canonical singularities, we see that ω_X is isomorphic to $\mathcal{O}_X(K_X)$. Since L is nef and A is ample, there exists a positive integer k_0 such that for every $k \geq k_0$, $(mL + kA) \otimes E$ admits a C^∞ -hermitian metric with (strictly) Nakano positive curvature. Then by the L^2 -vanishing theorem, we see that

$$H^q(X, \mathcal{O}_X(K_X + mL + kA) \otimes E) = 0$$

holds for every $m \geq 0$ and $q \geq 1$. This completes the proof. **Q.E.D.**

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Author's address
 Hajime Tsuji
 Department of Mathematics
 Sophia University
 7-1 Kioicho, Chiyoda-ku 102-8554
 Japan
 e-mail address: tsuji@mm.sophia.ac.jp